Math 210

Quiz # 1 , March 18, 2006

1. (10 pts.) If $\lim_{n\to\infty} a_n = a$, prove that $\{a_n\}$ is bounded.

2. (20 pts.) A sequence a_n is defined by

$$a_1 = 1, a_{n+1} = a_n + \frac{1}{\sqrt[3]{n}\, a_n}.$$

Obtain the asymptotic behaviour of a_n as follows: write, for a function g, an analogue of the given recurrence; find g, and use your answer to guess the asymptotic behaviour of a_n. Finally, and most importantly, prove your answer.

3. (10 pts.) $x, y > 0$ and $x \geq y$, find $\lim_{n\to\infty}(\frac{2x^n+7y^n}{4})^{1/n}$ and prove your answer.

4. (20 pts.) A sequence x_n is bounded by 2 and satisfies the inequalities

$$|x_{n+2} - x_{n+1}| \leq \frac{|x_{n+1}^2 - x_n^2|}{8}, n \in N, x_1, x_2 > 0.$$

Prove that x_n is a convergent sequence.

5. (10 pts.) Let G be a non-empty open subset of R, and let $p \in G$. Let $E = G^c \cap (-\infty, p]$, where G^c is the complement of G in R, and $(-\infty, p] = \{x : x \leq p\}$. Put $\alpha = \sup E$. Prove that α exists, and that $(\alpha, p] \subset G$. Prove also that $\alpha \notin G$.

6. (30 pts.) (a) If $b > a > 0$ and n is a positive integer, prove that

$$0 < (n+1)(b-a)a^n < b^{n+1} - a^{n+1} < (n+1)(b-a)b^n.$$

(b) Let h be the function defined by

$$h(x) = \lim_{n\to\infty}(1 + \frac{x}{n})^n$$

where the limit is assumed to exist for all real x.[This has already been done in class, so you do not need to repeat it].Using (a), prove that, for $x > 0$,

$$x \leq h(x) - 1 \leq xh(x).$$

(c) Prove that $h(x+y) = h(x)h(y)$, for all real x, y..